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Reciprocity theorem on generalized thermoelasticity based on parabolic heat conduction

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Abstract: In this research, we propose an innovative thermoelasticity theory that extends the work of Tzou [15] on parabolic-type dual-phase lag thermoelastic models. Our approach begins with the development of essential governing equations applicable to homogeneous and isotropic materials. These foundational equations serve as a springboard for the derivation of several significant theorems. The cornerstone of our study is a newly established reciprocity theorem, which we formulate using advanced Laplace transform methodology. This pivotal theorem offers profound insights into the complex interactions among diverse system components, thereby enhancing our understanding of thermoelastic phenomena.

Keywords: Generalized thermoelasticity; Dual-phase-lags; Reciprocal principle; Fourier’s law.

1. Introduction

In classical uncoupled thermoelasticity, mechanical and thermal fields do not interact, meaning that changes in elasticity do not influence temperature, and vice versa. This creates a scenario where Fourier's law leads to a parabolic heat equation, suggesting an infinite speed of thermal signal propagation, which presents an ill-posed problem in practical applications. This paradox has garnered significant interest among researchers in recent decades. To address these shortcomings, several scholars have advanced the theory of generalized thermoelasticity by modifying Fourier's law. The inception of classical coupled thermoelasticity theory was attributed to Biot [1], who introduced the exchange of mechanical and thermal energy, culminating in fundamentally parabolic-hyperbolic equations that also implied infinite thermal wave propagation speeds. A systematic approach to resolving the paradox in Biot's theory has been proposed by a number of notable authors [2-8], recognizing the necessity for finite thermal wave speed, often referred to as the second sound theory. The earliest work in this vein was conducted by Fox [9], who established a second sound theory based on continuum thermodynamics principles. Subsequently, Lord and Shulman [10] expanded upon this by introducing a model with a single thermal relaxation time parameter, while Green and Lindsay [11] further developed the theory by incorporating two thermal relaxation times, rooted in temperature-rate-dependent thermoelastic principles. Later, Green and Naghdi [12-14] proposed a more cohesive framework for heat propagation that included thermal pulse transmission. This evolution of thermoelasticity theory is categorized into three types: Type-I, Type-II, and Type-III. In this context, thermal displacement gradients and temperature gradients serve as constitutive variables. The Type-III model further generalizes the previous types by integrating a modified Fourier law of heat conduction, introducing a new constitutive variable known as thermal displacement. The study of short pulse waves and ultrafast heat transfer gained momentum in the twentieth century. To investigate microstructural heat transfer mechanisms at low temperatures, Tzou [15] proposed a dual-phase lag model of heat conduction, necessitating the introduction of two distinct phase lags with temperature gradients and heat flux vectors incorporated into Fourier's law. Subsequently, Quintanilla and Racke [16] explored the continuity and boundedness of solutions to the dual-phase lag heat conduction equation, while Horgan and Quintanilla [17] examined the spatial behavior of these solutions. Chandrasekharaiah [18] contributed to this discourse by analyzing a parabolic and hyperbolic model in relation to Tzou's dual-phase lag theory. Additionally, Roychoudhuri [19] studied an elastic half-space disturbance problem using a dual-phase thermoelastic model, thereby broadening the understanding of these complex interactions.

In the realm of linear isothermal elasticity, Betti's reciprocity theorem serves as a crucial framework for analyzing a body's deformation in response to specific surface tractions and body forces. Particularly, it allows for the assessment of deformation resulting from one set of forces when another set of forces is prescribed. This theorem becomes an invaluable asset in integrating the equations of elasticity through the employment of Green's function. Noteworthy contributions to the application of the reciprocity theorem in engineering problems have been made by Nowacki [20], who outlined various innovative uses, building upon the foundational works of Somigliana and Green. Furthermore, Khomyakevich and Rudenko [21] introduced an alternative method related to the theorem, expanding its applicability. Numerous researchers, including Predeleanu [22], Ionescu-Cazimir [23], Nowacki [20], Shivay, and Mukhopadhyay [24], Jangid, and Mukhopadhyay [25], have rigorously validated the reciprocity theorem within both anisotropic and isotropic homogeneous elastic media. Their findings underscore the theorem's versatility and significance in the field of elasticity theory.

This study establishes a reciprocal principle analogous to Betti's theorem. Our analysis focuses on fundamental equations subject to mixed boundary-initial value problems. Notably, we consider non-homogeneous initial conditions in our formulation. This theorem extends the scope and versatility of classical thermoelastic theorems, offering enhanced insights into thermal and mechanical field interactions.

This principle has potential applications in various fields of engineering and physics, particularly in areas such as structural mechanics, heat transfer, and wave propagation.

2. Basic governing equations

Equation of motion:

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i \tag{1}$$

Symmetry relation of stress tensor:

$$\sigma_{ij} = \sigma_{ji} \tag{2}$$

Entropy equation:

$$Q - \rho\theta_0 \dot{S} = q_{i,i} \tag{3}$$

Constitutive relation:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma\theta\delta_{ij} \tag{4}$$

$$q_i = -\mathrm{K}\theta_{,i} \tag{5}$$

$$\rho\theta_0 S = \rho c_E \theta + \theta_0 \gamma e_{kk} \tag{6}$$

Geometrical equations:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{7}$$

In the above equations, e_{ij} is the strain tensor, σ_{ij} is the stress tensor, u_i is the component of displacements, F_i is the component of the body force vector, θ is the absolute temperature referred from temperature θ_0, Q is the heat source per unit volume, q_i is the component of the heat flux vector, λ and μ are the Lame's elastic constants, ρ is the mass density, S is the entropy per unit mass, K is the thermal conductivity of the material, K^* the rate of thermal conductivity, c_E is the specific heat at constant strain, $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t is the linear thermal expansion coefficient, e is the dilatation and ∇^2 is the Laplacian operator. Suppose that the stresses and strains are functions of class C^1 whereas the displacements and temperatures are functions of class C^2 for all $x \in V + \partial V$.

The heat conduction equation under thermoelasticity with dual-phase-lags of parabolic type [15];

$$\left[\mathrm{K}^* + \mathrm{K}\left(1 + \zeta_T \frac{\partial}{\partial t}\right)\right]\nabla^2\theta = \left(1 + \zeta_q \frac{\partial}{\partial t}\right)\left(\rho c_v \dot{\theta} + \gamma\theta_0 \dot{u}_{i,i} - Q\right) \tag{8}$$

Here ζ_T the temperature gradient and ζ_q the heat flux are the phase lags such that $\zeta_T < \zeta_q$.

From equations. (1), (4), and (7), we have

$$\lambda u_{j,ji} + \mu\left(u_{i,ji} + u_{j,ji}\right) - \gamma\theta_{,i} + F_i = \rho\ddot{u}_i \tag{9}$$

For the system of field equations described above, consider homogeneous initial conditions as

$$u_i(x,0) = 0,\ \dot{u}_i(x,0) = 0;\ x \in V \quad \text{and} \quad \theta(x,t) = 0, \quad \dot{\theta}(x,t) = 0; \quad x \in \partial V \tag{10}$$

and the prescribed boundary conditions as

$$\sigma_i(x,t) = \sigma_{ji}n_j = p_i(x,t); \quad \theta(x,t) = \theta_0(x,t); \quad x \in \partial V, \quad t > 0 \tag{11}$$

where p_i are surface traction components and temperature $\theta_0(x,t)$ is prescribed on the boundary surface ∂V of the solid.

3. Reciprocity theorem for DPL Parabolic

In initial-boundary value problems presented in equations (1)-(11), two distinct thermoelatstic loadings systems are applied to a homogeneous, isotropic bounded thermoelastic body given as

$$\Omega^{\alpha} = (F_i^{\alpha}, \quad p_i^{\alpha} : Q^{\alpha}, \quad \theta_0^{\alpha}), \quad \alpha = 1, 2 \tag{12}$$

where, $F_i, p_i, Q,$ and θ_0 are the body forces, surface traction, heat source, and surface heating respectively.

Also, the two corresponding thermoelastic configurations are given below.

$$I^{\alpha} = (u^{\alpha}, \quad \theta^{\alpha}), \ \alpha = 1, 2 \tag{13}$$

Now for $\alpha, \beta = 1, 2$. Suppose

$$\begin{aligned}
\Omega_{\alpha\beta} = \mathrm{K}^* \int_{\partial V} dA(x) \int_0^t \left[\theta_0^{(\beta)}(x,\tau)\theta_{,n}^{(\alpha)}(x,t-\tau)\right] d\tau \\
+ \mathrm{K} \int_{\partial V} dA(x) \int_0^t \left[\theta_0^{(\beta)}(x,\tau)\left(1 + \zeta_T \frac{\partial}{\partial\tau}\right)\theta_{,n}^{(\alpha)}(x,t-\tau)\right] d\tau \\
+ \Bigg[\int_{\partial V} dV(x) \int_0^t Q^{(\alpha)}(x,\tau)\left(1 + \zeta_q \frac{\partial}{\partial\tau}\right)\theta^{(\beta)}(x,t-\tau)\, d\tau \\
+ \theta_0 \int_{\partial V} dA(x) \int_0^t p_i^{(\beta)}(x,t-\tau)\frac{\partial}{\partial\tau}\left(1 + \zeta_q \frac{\partial}{\partial\tau}\right) u_i^{(\alpha)}(x,\tau)\, d\tau \\
+ \theta_0 \int_V dV(x) \int_0^t F_i^{(\beta)}(x,t-\tau)\frac{\partial}{\partial\tau}\left(1 + \zeta_q \frac{\partial}{\partial\tau}\right) u_i^{(\alpha)}(x,\tau)\, d\tau\Bigg]
\end{aligned} \tag{14}$$

where $\theta_{,n} = \theta_{,i}$ denotes the derivative of the temperature θ along normal to the surface ∂V.

Then, $$\Omega_{12} = \Omega_{21} \tag{15}$$

Proof: - By hypothesis, apply loading in equation (4), we get

$$\sigma_{ij}^{(\alpha)} = \left(\lambda e_{kk}^{(\alpha)} - \gamma\theta^{(\alpha)}\right) + 2\mu e_{ij}^{(\alpha)} \ , \alpha = 1, 2 \qquad (16)$$

Taking Laplace to transform over the boundary condition (11) and the equation (17) defined by

$$\tilde{g}(x, y, s) = \mathcal{L}[g(x, y, t)] = \int_0^\infty g(x, y, t)e^{-st}dt, \quad s > 0$$

Therefore, we obtain (for $\alpha = 1, 2$)

$$\tilde{p}_i(x, t) = \tilde{\sigma}_{ji}n_j; \quad \tilde{\theta}(x, t) = \tilde{\theta}_0(x, t); \ x \in \partial V, \qquad t > 0 \qquad (17)$$

$$\tilde{\sigma}_{ij}^{(\alpha)} = 2\mu\tilde{e}_{ij}^{(\alpha)} + \left[\lambda\tilde{e}_{kk}^{(\alpha)} - \gamma\tilde{\theta}^{(\alpha)}\right]\delta_{ij} \qquad (18)$$

Multiplying (18) for $\alpha = 1$ by $\tilde{e}_{ij}^{(2)}$ and for $\alpha = 2$ by $\tilde{e}_{ij}^{(1)}$. Take volume integral after subtracting the results and using the relation

$$\left(2\mu\tilde{e}_{ij}^{(1)} + \lambda\tilde{e}_{kk}^{(1)}\delta_{ij}\right)\tilde{e}_{ij}^{(2)} = \left(2\mu\tilde{e}_{ij}^{(2)} + \lambda\tilde{e}_{kk}^{(2)}\delta_{ij}\right)\tilde{e}_{ij}^{(1)}$$

We obtain the relation

$$\int_V \left(\tilde{\sigma}_{ij}^{(1)}\tilde{e}_{ij}^{(2)} - \tilde{\sigma}_{ij}^{(2)}\tilde{e}_{ij}^{(1)}\right)dV = \gamma \int_V \left(\tilde{\theta}^{(2)}\tilde{e}_{kk}^{(1)} - \tilde{\theta}^{(1)}\tilde{e}_{kk}^{(2)}\right) dV \qquad (19)$$

Now applying the Laplace transform of equation (7), we have from equation (19)

$$\int_V \left(\tilde{\sigma}_{ij}^{(1)}\tilde{u}_{i,j}^{(2)} - \tilde{\sigma}_{ij}^{(2)}\tilde{u}_{i,j}^{(1)}\right)dV = \gamma \int_V \left(\tilde{\theta}^{(2)}\tilde{e}_{kk}^{(1)} - \tilde{\theta}^{(1)}\tilde{e}_{kk}^{(2)}\right) dV \qquad (20)$$

Taking Laplace transform on equation (1) and employing the initial homogeneous conditions (10), we get

$$\tilde{\sigma}_{ij,j} + \tilde{F}_i = \rho s^2\tilde{u}_i \ ; \quad x \in V \qquad (21)$$

Applying Gauss's divergence theorem over the LHS of equation (20) and use the equation (21) to obtain the result as

$$\int_{\partial V} \left(\tilde{p}_i^{(1)}\tilde{u}_i^{(2)} - \tilde{p}_i^{(2)}\tilde{u}_i^{(1)}\right) dA + \int_V \left(\tilde{F}_i^{(1)}\tilde{u}_i^{(2)} - \tilde{F}_i^{(2)}\tilde{u}_i^{(1)}\right) dV + \gamma \int_V \left(\tilde{\theta}^{(1)}\tilde{e}_{kk}^{(2)} - \tilde{\theta}^{(2)}\tilde{e}_{kk}^{(1)}\right) dV = 0 \qquad (22)$$

The cause of a mechanical nature is only contained in equation (22): the force exerted by the body and the surface's traction.

Next, we examine thermoelastic configurations adjoining the heat conduction equation

$$I^{(\alpha)} = \left(u^{(\alpha)}, \theta^{(\alpha)}\right); \quad \alpha = 1, 2$$

Now, applying Laplace transform over the equation (9), we find the following on simplifying

$$\nabla^2\tilde{\theta}^{(\alpha)} = \frac{(1+\zeta_q s)\rho c_v s}{\mathrm{K}(1+\zeta_T s)}\tilde{T}^{(\alpha)} + \frac{(1+\zeta_q s)\gamma\theta_0 s}{\mathrm{K}(1+\zeta_T s)}\tilde{e}_{kk}^{(\alpha)} - \frac{(1+\zeta_q s)}{\mathrm{K}(1+\zeta_T s)}\tilde{Q}^{(\alpha)} \quad ;\alpha = 1,2 \tag{23}$$

Multiplying (23) for $\alpha = 1$ by $\theta^{(2)}$ and for $\alpha = 2$ by $\tilde{\theta}^{(1)}$, taking volume integral after subtracting the results we find the identity as

$$\int_V \left(\nabla^2\tilde{\theta}^{(2)}\tilde{\theta}^{(1)} - \nabla^2\tilde{\theta}^{(1)}\tilde{\theta}^{(2)}\right)dV = \frac{(1+\zeta_q s)\gamma\theta_0 s}{\mathrm{K}(1+\zeta_T s)}\int_V\left(\tilde{\theta}^{(1)}\tilde{e}_{kk}^{(2)} - \tilde{\theta}^{(2)}\tilde{e}_{kk}^{(1)}\right)dV + \frac{(1+\zeta_q s)}{\mathrm{K}(1+\zeta_T s)}\int_V\left(\tilde{Q}^{(1)}\tilde{\theta}^{(2)} - \tilde{Q}^{(2)}\tilde{\theta}^{(1)}\right)dV \tag{24}$$

Again, applying Gauss's divergence theorem over LHS of equation (24) and we get

$$\int_{\partial V}\left(\tilde{\theta}_{,n}^{(2)}\tilde{\theta}_0^{(1)} - \tilde{\theta}_{,n}^{(1)}\tilde{\theta}_0^{(2)}\right)dA = \frac{(1+\zeta_q s)\gamma\theta_0 s}{\mathrm{K}(1+\zeta_T s)}\int_V\left(\tilde{\theta}^{(1)}\tilde{e}_{kk}^{(2)} - \tilde{\theta}^{(2)}\tilde{e}_{kk}^{(1)}\right)dV + \frac{(1+\zeta_q s)}{\mathrm{K}(1+\zeta_T s)}\int_V\left(\tilde{Q}^{(1)}\tilde{\theta}^{(2)} - \tilde{Q}^{(2)}\tilde{\theta}^{(1)}\right)dV \tag{25}$$

Equation (25) can be rewritten in the form

$$\int_V\left(\tilde{\theta}^{(1)}\tilde{e}_{kk}^{(2)} - \tilde{\theta}^{(2)}\tilde{e}_{kk}^{(1)}\right)dV = \frac{\mathrm{K}(1+\zeta_T s)}{\gamma\theta_0 s(1+\zeta_q s)}\int_{\partial V}\left(\tilde{\theta}_{,n}^{(2)}\tilde{\theta}_0^{(1)} - \tilde{\theta}_{,n}^{(1)}\tilde{\theta}_0^{(2)}\right)dA + \frac{1}{\gamma\theta_0 s}\int_V\left(\tilde{Q}^{(1)}\tilde{\theta}^{(2)} - \tilde{Q}^{(2)}\tilde{\theta}^{(1)}\right)dV \tag{26}$$

Using equation (26) in equation (22) we finally obtain;

$$\mathrm{K}(1+\zeta_T s)\int_{\partial V}\left(\tilde{\theta}_{,n}^{(1)}\tilde{\theta}_0^{(2)} - \tilde{\theta}_{,n}^{(2)}\tilde{\theta}_0^{(1)}\right)dA + (1+\zeta_q s)\int_V\left(\tilde{Q}^{(1)}\tilde{\theta}^{(2)} - \tilde{Q}^{(2)}\tilde{\theta}^{(1)}\right)dV = \theta_0 s(1+\zeta_q s)\left[\int_{\partial V}\left(\tilde{p}_i^{(1)}\tilde{u}_i^{(2)} - \tilde{p}_i^{(2)}\tilde{u}_i^{(1)}\right)dA + \int_V\left(\tilde{F}_i^{(1)}\tilde{u}_i^{(2)} - \tilde{F}_i^{(2)}\tilde{u}_i^{(1)}\right)dV\right] \tag{27}$$

Applying the convolution theorem given below for taking inverse Laplace transforms in equation (27)

$$\mathcal{L}^{-1}\big(\tilde{g}_1(s)\tilde{g}_2(s)\big) = \int_0^t g_1(\tau)\, g_2(t-\tau)d\tau = \int_0^t g_2\,(\tau) g_1(t-\tau)d\tau$$

Thus we get equation (14) with the use of the convolution theorem on equation (27).

It concludes that the reciprocity theorem has been proven for the dual-phase-lags hyperbolic model of generalized thermoelasticity.

4. Conclusion

In this study, we establish a reciprocal principle analogous to Betti's theorem within the context of thermoelasticity. Our analysis focuses on fundamental equations subject to mixed boundary-initial value problems in thermoelastic media. Notably, we consider non-homogeneous initial conditions in our formulation, extending the applicability of the reciprocal theorem to a broader range of practical scenarios.

Detailed applications of thermoelasticity are mentioned as below:

1. Structural Engineering
   i. Calculate displacements and stresses in complex structures under non-uniform temperature distributions
   ii. Analyze thermal bridges in building envelopes for improved energy efficiency
2. Heat Transfer Engineering
   i. Determine temperature fields in deforming materials
   ii. Optimize heat sink designs for electronic components, considering both thermal and mechanical effects
3. Stress Analysis
   i. Calculate thermal stresses in composite materials with varying thermal expansion coefficients
   ii. Predict and mitigate thermal fatigue in aerospace structures
4. Thermal Protection Systems
   i. Optimize heat shield designs for spacecraft re-entry
   ii. Analyze performance of thermal barrier coatings in gas turbines
5. Smart Materials and Structures
   i. Analyze shape memory alloys' response to thermal-mechanical loads
   ii. Design piezoelectric sensors and actuators for variable temperature environments
6. Geomechanics
   i. Study thermally induced rock deformations in geothermal energy extraction

ii. Assess stability of underground structures subject to thermal loads

7. Biomechanics
   i. Model heat transfer and mechanical stress in artificial joints
   ii. Investigate temperature effects on biological tissues
8. Manufacturing Processes
   i. Analyze residual stresses in welding and heat treatment processes
   ii. Optimize 3D printing parameters to minimize thermal distortion
9. Energy Storage Systems
   i. Design more efficient and durable batteries by accounting for thermal expansion and contraction during charge-discharge cycles
10. Nanomechanics
   i. Study thermal transport and mechanical behavior in nanostructures
   ii. Develop accurate models for nanoelectromechanical systems (NEMS) operating under varying thermal conditions

This reciprocal theorem provides a powerful analytical tool for solving complex thermoelastic problems, offering insights into the coupled nature of thermal and mechanical effects in materials across various scales and applications.

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Conflict of interest

There are no potential conflicts among the authors.

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